Large Deflections Analysis of Thin Cantilever Beams Using Numerical Integration and Experimental Procedures

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Abstract

This paper deals with the large deflection of thin cantilever beams of rectangular cross-section subjected to a lateral concentrated load at the free end. Because of the large deflection, geometric nonlinearity arises and, therefore, the analysis was formulated according to the non-linear bending theory in which the squares of the first derivatives of the governing Bernoulli-Euler equation cannot be neglected as is done in classical beam theory. The resulting second-order non-linear differential equation leading to elliptical function was solved using the Gauss-Legendre numerical integration method of two to six points approach and are presented in graphical form. Simple experimental procedures were performed for validation purpose. Beams having deep to width ratio of 0.13 and 0.27 were used. It was found in general that experimental results in terms of rotation, vertical deflection, and horizontal deflection of the end point of the beams were bigger than those obtained by numerical integration for every deep to width ratio, h/b, where the highest differences were found for high value of h/b. The differences would be expected from the material nonlinear effects. It should be noted that material nonlinearity was not considered in the numerical integration, while this effect might be present in the experimental procedures

Keywords: large deflections, cantilever beams, numerical integration, experimental procedures

Introduction

Flexible structures have been used in many real-world machines. For example, leaf springs are used in the suspension systems of cars. Helical springs are used in the shock absorbers of racing motorcycles. Flexible metal strips are used in controlling the arm-type positioning of mechanisms of magnetic disk driver of computers. Transmission cables are very often subjected to large deflection. Helicopter rotor blades and wind turbine blades are types of flexible cantilever beam. Such flexible structures can undergo large deflections and rotations without exceeding their elastic limits.

The subject of the large deflection of cantilever beams is one of great historic interest. Thin cantilever beams, being flexible, exhibit large deflections and slopes when subjected to loads. Due to these large deflections, geometric nonlinearity arises while strains remain small. As a result, problems involving large deflections of cantilever beams must be formulated according to the non-linear bending theory. In such, the squares of the first derivatives in the governing Bernoulli-Euler equation cannot be neglected as is done classical beam theory. In addition, correction factors for shortening the moment arm become the major contribution to the solution of large deflection problems.

The solution for the large deflection of a cantilever beam of linear elastic material subjected to a lateral load at the free end was first obtained by [Barten, 1944]. He was able to show that the difference between the deflection as found by the classical beam theory and the large deflection theory can be noticed only in the case of beams of low stiffness. Bisshopp and Drucker [1945] derived the solution for large deflection of cantilever beams from the fundamental Bernoulli-Euler theorem which states that the curvature is proportional to the bending moment. In their derivation, it was assumed that the length of the beam is inextensible. Bisshopp [1973] derived some rational approximation for the end slope and deflection of a cantilever beam by linearization of the elliptic integral solution (by combining the first and second order approximation for small quantities with obvious properties of the deflected configuration). Rohde [1953] provides solutions to large deflection of a cantilever beam subjected to uniformly distributed load by expanding the slope in a power series of the arc length. Holden [1972] obtained numerical solution to large deflection problem of a cantilever beam under uniformly distributed load using a fourth-order Runge-Kutta method. Wang [1969] considered the large deflections of cantilever beam and provided solution for the slope in term of the horizontal projection of the arc length. Baker [1993] obtained the large deflection solution to linear elastic non-prismatic beam under arbitrary distributed loads through a weighted residual solution to the governing Bernoulli-Euler equation. Lee et al. [1993] investigated the large deflection of linear elastic cantilever beam of variable cross-section under combined loading by using Runge-Kutta-Fehlber method. Lewis and Monasa [1981] numerically solved the problem of large deflection of cantilever beams made of nonlinear elastic materials of the Ludwick type subjected to a concentrated load at the free end. In another paper, Lewis and Monasa [1982] reconsidered the problem of large deflection of cantilever beams made of Ludwick type material subjected to an end pure moment and obtained a closed-form solution. Lee [2002] investigated the large deflection of cantilever beam made of Ludwick type material under a combined loading of a uniformly distributed load and a lateral vertical load at the free end. Butchher's fifth order Runge-Kutta method was used to obtain slopes of any point along the arc length.

This paper deals with the large deflection of thin cantilever beams of rectangular cross-section subjected to a lateral concentrated load at the free end. The resulting second-order non-linear differential equation leading to elliptical function was solved using the Gauss-Legendre numerical integration method of two to six points approach to obtain results in terms of rotation, vertical deflection, and horizontal deflection of the end point of the beams. Simple experimental procedures were performed for validation purpose.

The Governing Elastic Curve

Figure 1 shows a cantilever beam subjected to lateral concentrated load, P, at its free end. The clamped end of the beam is taken as the origin of coordinate and downward deflections are considered as positive. A point on the beam is identified by four quantities of which only one is independent. These four quantities are the two rectangular coordinate x and y, the arc length s measured from the origin of the coordinate and the deflection angle θ which is the angle between the tangent to the curve at the point under consideration and the horizontal. This point may be identified by the symbol (x, y, s, θ). The beam has a uniform cross-section moment of inertia I and is composed of a material whose modulus of elasticity is E. Due to load P, the end point B will move to B', produce rotational displacement, θ_b, horizontal displacement, δ_h and vertical displacement, δ_v. If the large deflection effects are taken into account, the governing elastic curve can be written as:

$$\frac{\dfrac{d^2 v}{dx^2}}{\left[1+\left(\dfrac{dv}{dx}\right)^2\right]^{3/2}} = -\frac{M}{EI} \tag{1}$$

Equation (1) is known as Bernoulli-Euler equation for large deflection of a cantilever beam. For small deflection assumption, the square of the first derivative in equation (1) is neglected and reduced to the simple beam bending theory:

$$\frac{d^2 v}{dx^2} = -\frac{M}{EI} \tag{2}$$

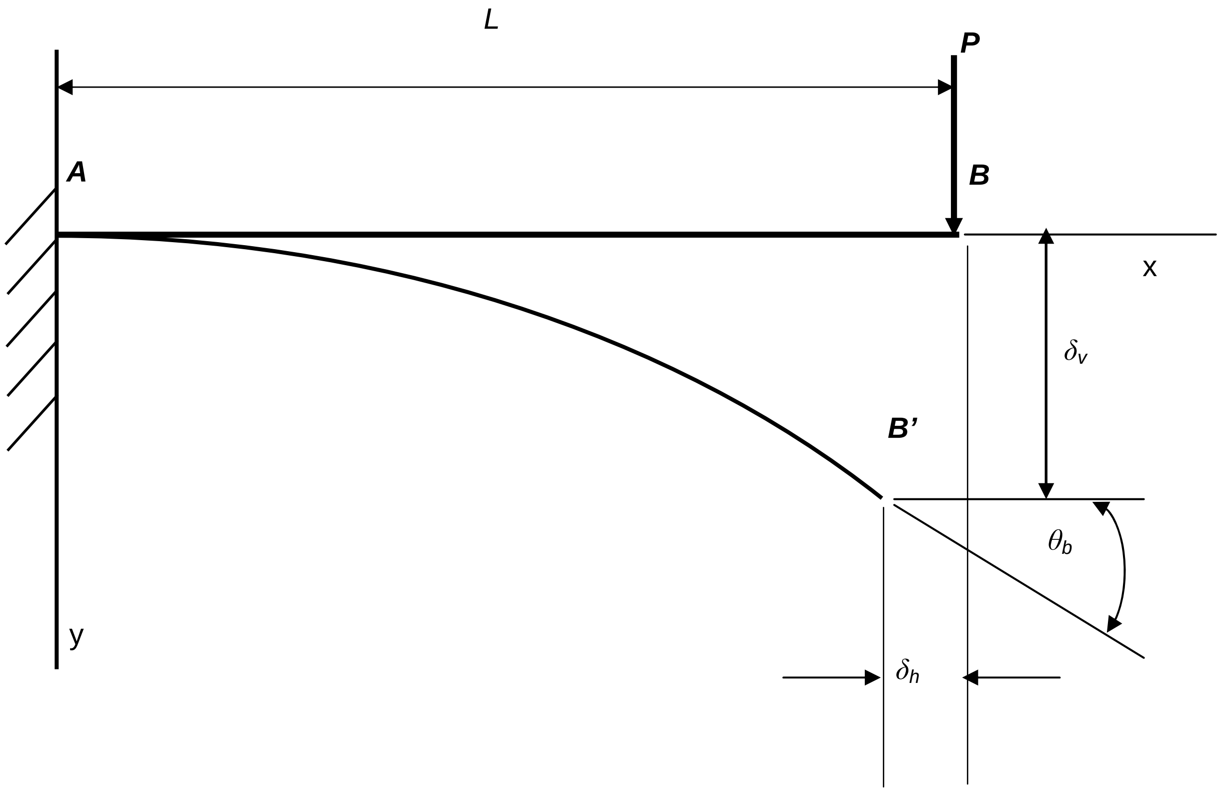


Figure 1 Large deflection of a cantilever beam under end concentrated load

Numerical Integration

The solution to equation (1) was provided in elliptical function [Bisshopp and Drucker, 1945]:

$$F(k) - F(k,\phi) = \sqrt{\frac{PL^2}{EI}} \tag{3}$$

Where,

$$k = \sqrt{\frac{1+\sin\theta_b}{2}} \tag{4}$$

$$\phi = arc\sin\frac{1}{k\sqrt{2}} \tag{5}$$

$$F(k) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2\sin^2 t}} \quad \text{; is complete elliptic integral of first kind} \tag{6}$$

$$F(k,\phi) = \int_0^{\phi} \frac{dt}{\sqrt{1-k^2 \sin^2 t}}$$; incomplete elliptic integral of first kind (7)

Vertical deflection per unit length can be written as:

$$\frac{\delta_v}{L} = 1 - \sqrt{\frac{4EI}{PL^2}}\left[E(k) - E(k,\phi)\right] \tag{8}$$

Where,

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2 \sin^2 t}\ dt$$; is complete elliptic integral of second kind (9)

$$E(k,\phi) = \int_0^{\phi} \sqrt{1-k^2 \sin^2 t}\ dt$$; is incomplete elliptic integral of second kind (10)

Finally, the horizontal deflection per unit length of the beam can be calculated from:

$$\frac{\delta_h}{L} = 1 - \sqrt{\frac{2EI \sin \theta_b}{PL^2}} \tag{11}$$

Obtaining the direct solution to equations (3) through (11) would be time consuming. In this paper solution were obtained using numerical integration (Gauss Quadrature). In general, the formula for Gauss-Legendre can be written as:

$$I \cong c_0 f(x_0) + c_1 f(x_1) + \ldots + c_{n-1} f(x_{n-1}) \tag{12}$$

Where, the value of the coefficient *c's* and the argument *x's* can be found in any textbook of numerical methods for engineers [see, for example, Chapra & Canale, 1998]

It should be noted that before equation (12) is employed, the lower and upper bounds of the integral must be first set for -1 and +1 respectively. The complete procedure of the numerical integration using Gauss-Quadrature can be found in [Tarigan, 2005].

Experimental Set-Up

Figure 2 shows the experimental set-up used in this analysis. It consists of horizontal deflection measurement devise A_1 and A_2, vertical deflection measurement devise A_3, A_4, cantilever flexible B of 800mm length, 15mm width, and 2mm, 3mm, and 4mm thick, supports C and D, load E, and Basement F.

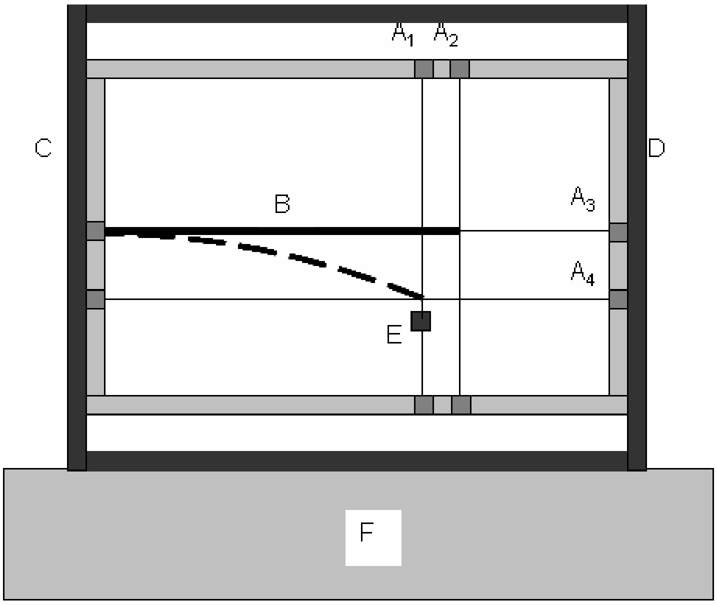


Figure 2 Experimental set-up for large deflection of a cantilever beam

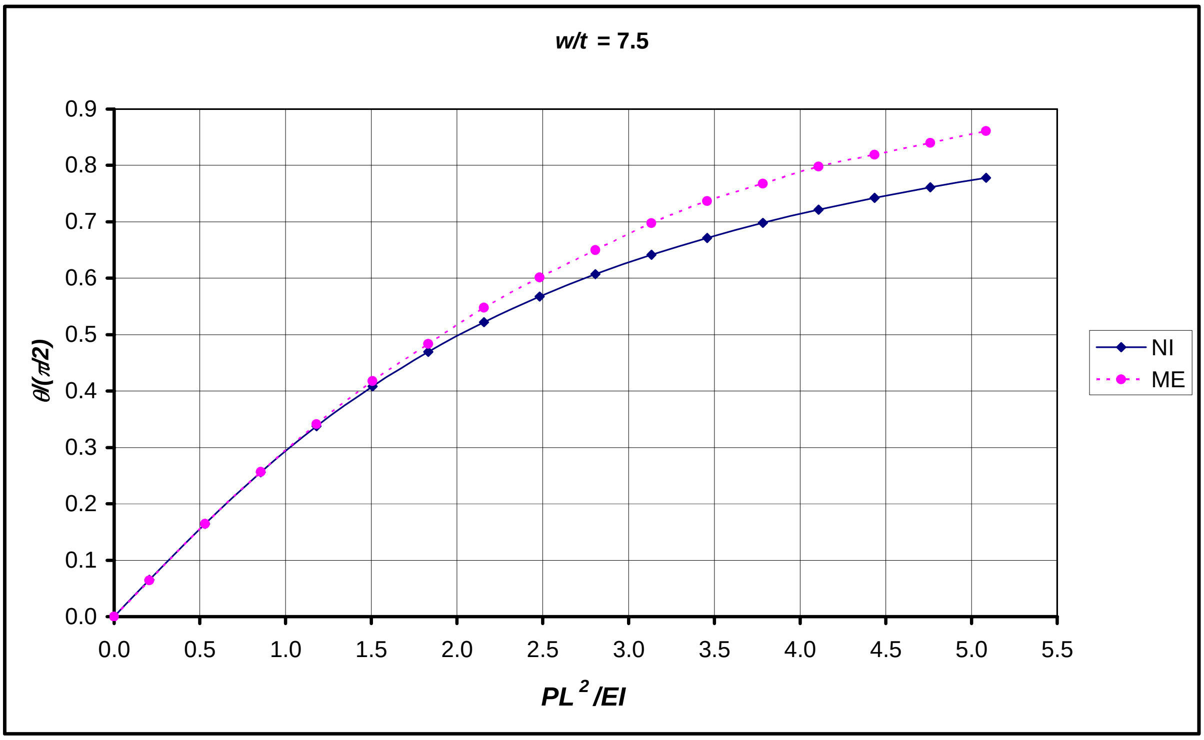
Results and Discussion

Figure 3(a) and (b) show plots of normalized end rotation, $\theta/(\pi/2)$ against non-dimensional parameter, $\frac{PL^2}{EI}$. It can be clearly seen that the end rotation obtained by experimental method is bigger than those by numerical integration for *w/t* equal 7.5 and 3.75. The nonlinear effect is more pronounced for lower value of *w/t*. It is a trend of nonlinear softening.

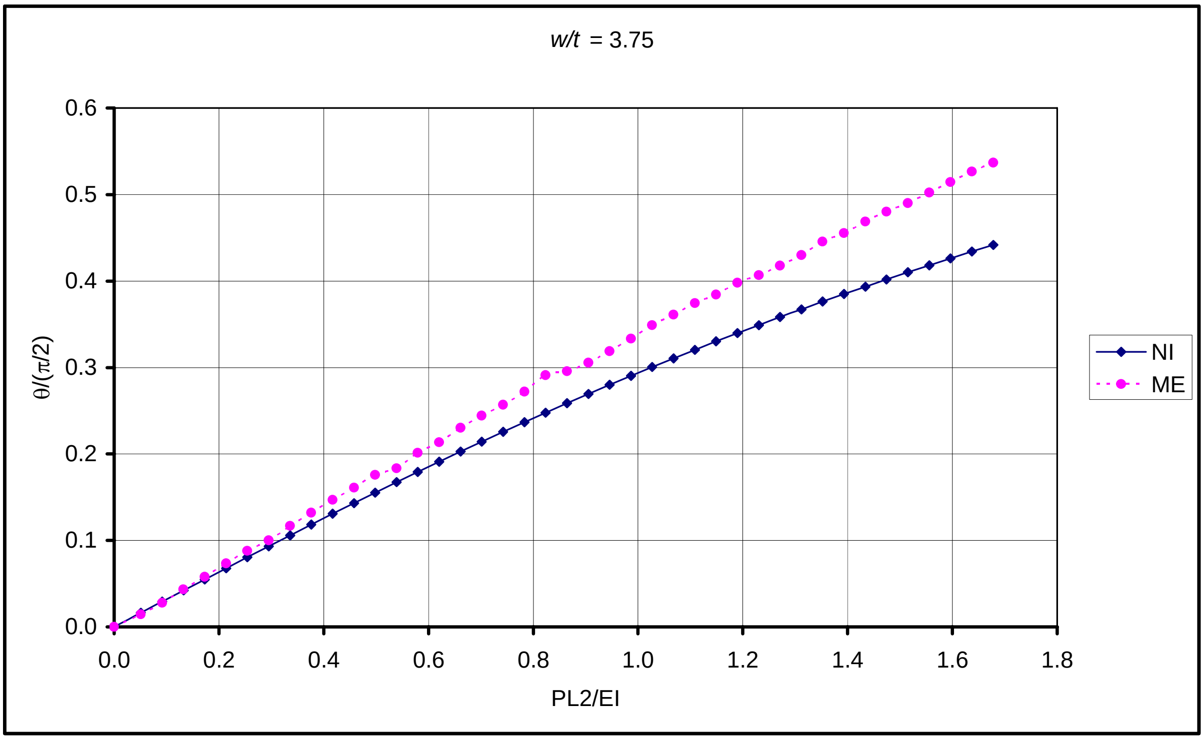
Figure 4(a) and b shows graphs for normalized vertical deflection, *dv/L*, against non-dimensional parameter, $\frac{PL^2}{EI}$. It can be seen that the vertical displacement of the loaded end obtained by experimental is bigger than those obtained by numerical integration for *w/t* equal 7.5 and 3.75. Again, the nonlinear effects are more pronounced for lower value of *w/t* and it a trend of nonlinear softening.

Figure 5(a) and b shows graphs for normalized horizontal displacement, *dh/L*, against non-dimensional parameter, $\frac{PL^2}{EI}$. Again, it can be seen that the horizontal deflection of the loaded end obtained by experimental is bigger than those obtained by numerical integration for *w/t* equal 7.5 and 3.75. Again, the nonlinear effects are more pronounced for lower value of *w/t*, but it is a nonlinear softening for *w/t* equal 7.5 and nonlinear hardening for *w/t* equal 3.75.

It is worth to note that the rotation, vertical deflection, and horizontal deflection of the loaded end obtained by experimental methods are bigger than those obtained by numerical integration for all values of width to thickness ratio of the beam cross-section. Possible cause is the material nonlinear effect. It should be noted that the material nonlinearity was not considered in the numerical integration, but this effect might be present in the experimental procedure.

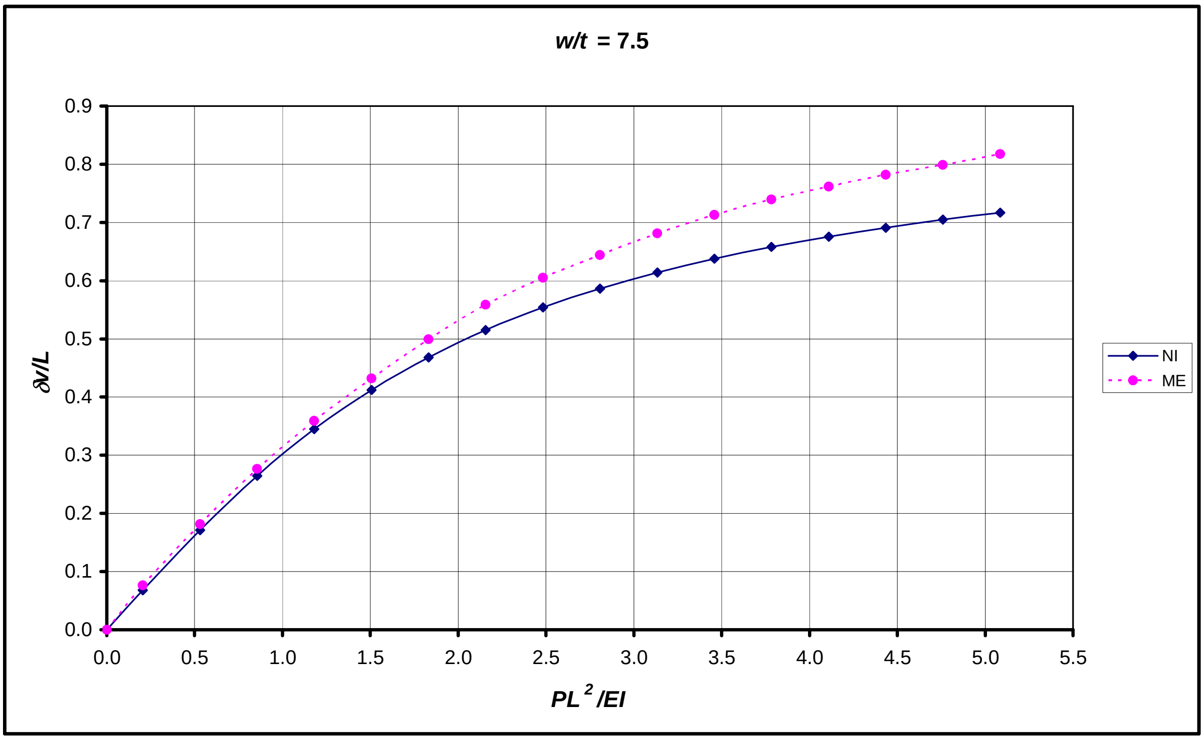


(a)

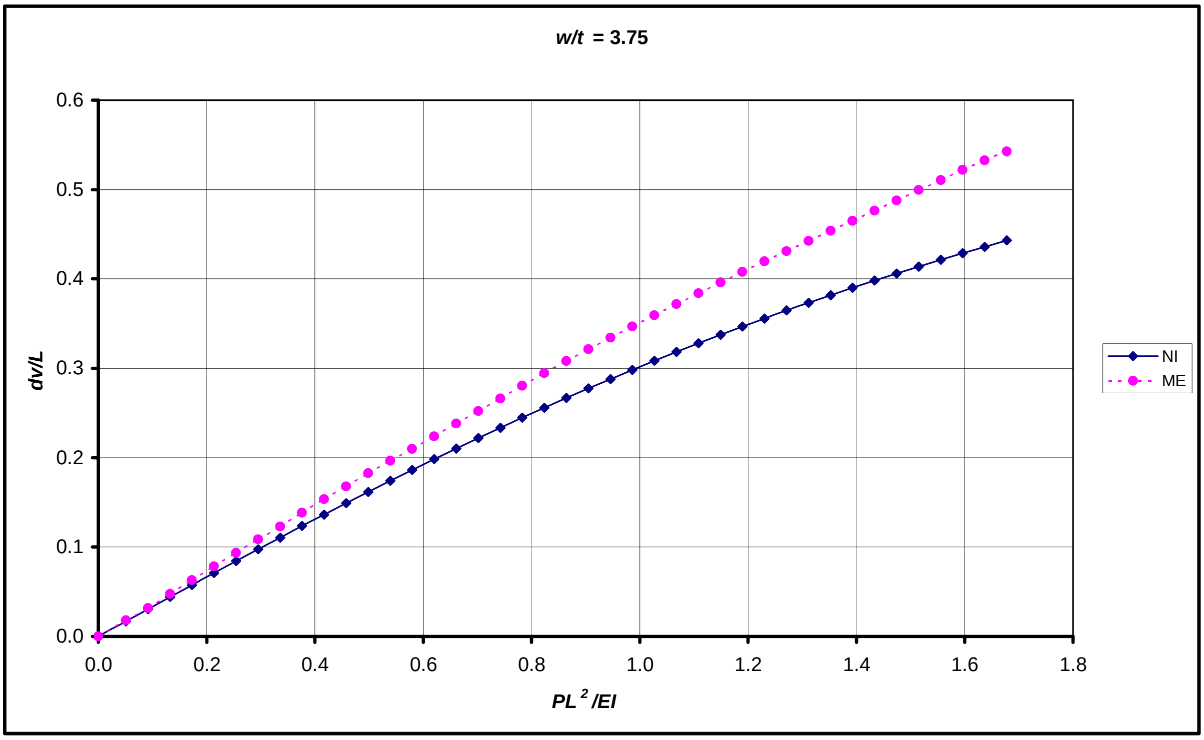


(b)

Figure 3 normalized rotation of the loaded end, (a) *w/t* = 7.5, (b) *w/t* = 3.75

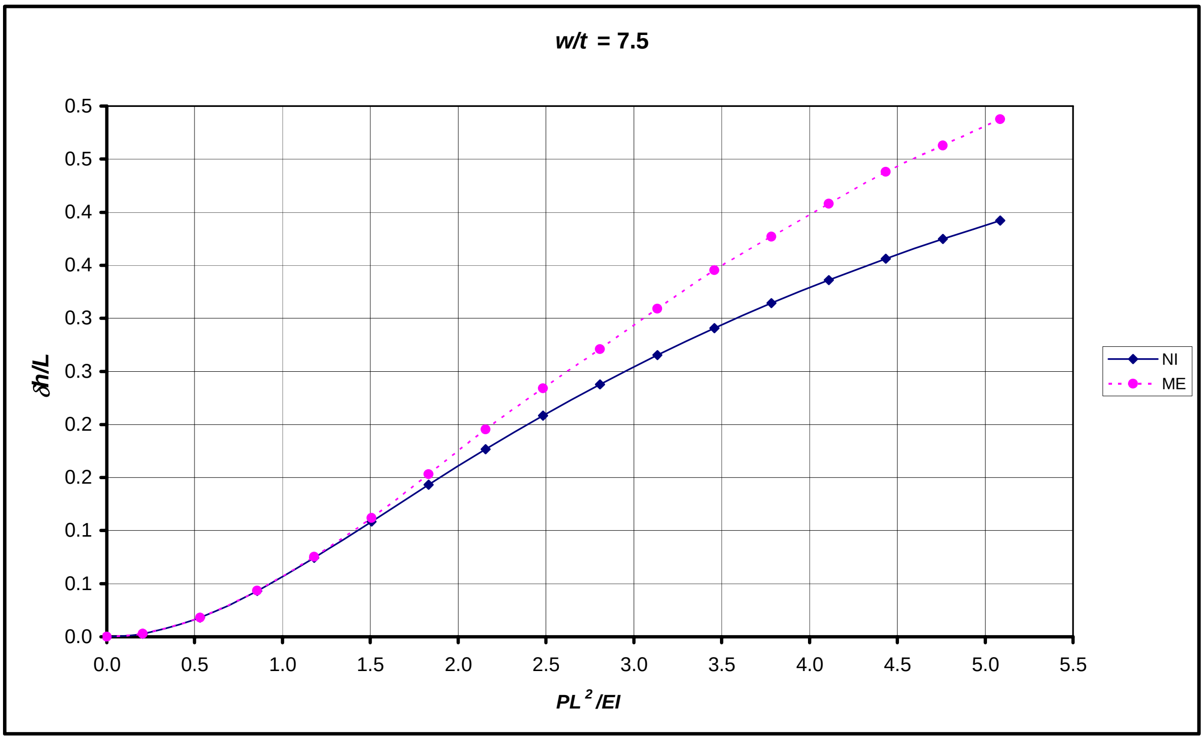


(a)

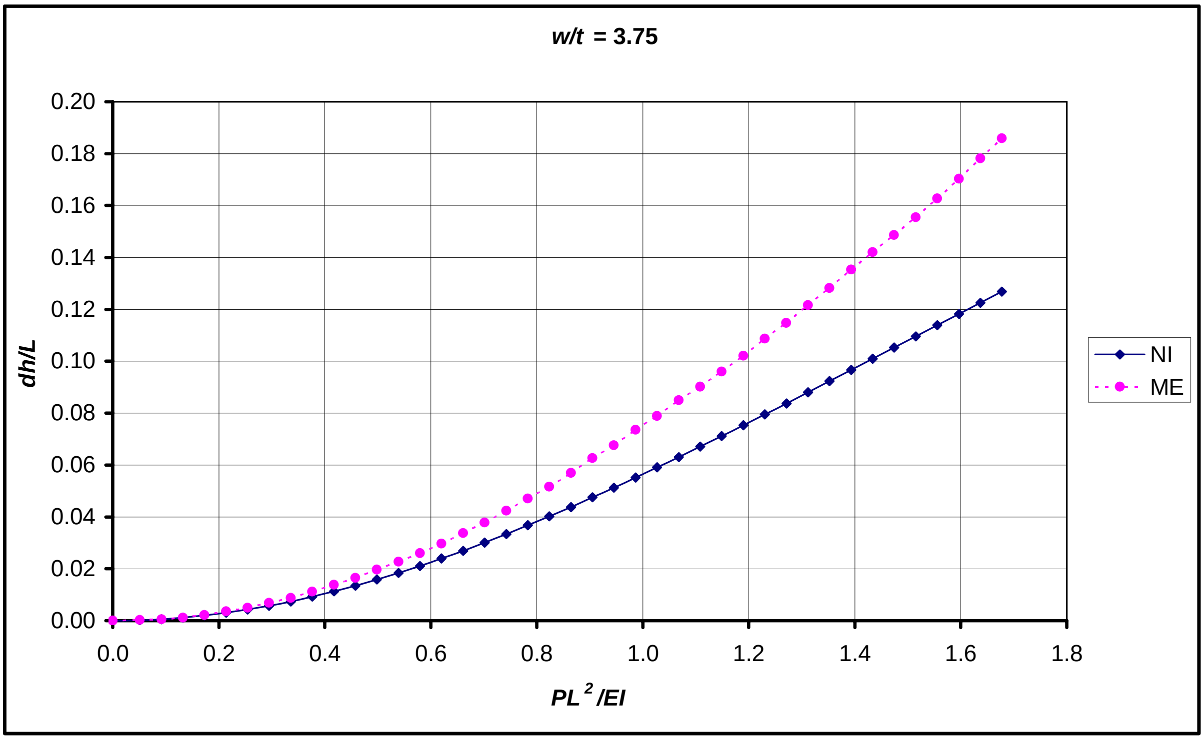


(b)

Figure 4 normalized vertical deflection of the loaded end, (a) *w/t* = 7.5, (b) *w/t* = 3.75



(a)



(b)

Figure 5 normalized horizontal deflection of the loaded end, (a) *w/t* = 7.5, (b) *w/t* = 3.75

Conclusion

Geometric nonlinearity of large deflection of cantilever beams was investigated. Rotation, vertical deflection, and horizontal deflection of the loaded end were obtained by numerical integration and experimental procedure. It can be concluded that the deflections of the loaded end obtained experimentally were bigger than those obtained numerically. The possible cause of their different is the material presence of nonlinear effect in experimental procedure, the effect neglected in the numerical integration.

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